Lavrentiev's phenomenon, approximation, and regularity

University of Warsaw, 20-22.11.2023

Book of abstracts

	Monday 20.11	Tuesday 21.11	Wednesday 22.11
09:15	registration		
09:30	Marcellini	Balci	Bulicek
10:15	Koch	Borowski	Mosconi
11:00	coffee break	coffee break	coffee break
11:30	Leonetti	De Filippis	Molchanova
12:15	Hasto	Ruf	Antonini
13:00	lunch break	lunch break	
14:45	Cianchi	Treu	
15:30	discussions with coffee	discussions with coffee	

Localization:
Faculty of Mathematics, Informatics and Mechanics
Banacha 2a, Warsaw
Room 2180

Second-order estimates in anisotropic elliptic problems

Carlo Alberto Antonini
University of Milan, Italy

In recent years, various results showed that second-order regularity of solutions to the p-Laplace equation can be properly formulated in terms of the expression under the divergence, the so-called stress field.
I will discuss the extension of these results to the anisotropic p-Laplace problem, namely equations of the kind $-\text{div}(\mathcal{A}(\nabla u)) = f$, in which the stress field is given by $\mathcal{A}(\nabla u) = H^{p-1}(\nabla u)\nabla_\xi H(\nabla u)$, where $H(\xi)$ is a norm on $\mathbb{R}^n$ satisfying suitable ellipticity assumptions.
$W^{1,2}$-Sobolev regularity of $\mathcal{A}(\nabla u)$ is established when f is square integrable, and both local and global estimates are obtained. The latter apply to solutions to homogeneous Dirichlet problems on convex domains. A key point in our proof is an extension of Reilly's identity to the anisotropic setting.
This is joint work with A. Cianchi, G. Ciraolo, A. Farina and V.G. Maz'ya.

C.A. Antonini, G. Ciraolo, A. Farina, *Interior regularity results for inhomogeneous anisotropic quasilinear equations*, Math. Ann. (2022).

C.A. Antonini, A. Cianchi, G. Ciraolo, A. Farina, V.G. Maz'ya, *Global second-order estimates in anisotropic elliptic problems*, arXiv preprint (2023) arXiv:2307.03052.

Nonlocal and mixed models in presence of energy gap

Anna Balci
Charles University in Prague, Czech Republic

The essential feature of many models with non-standard growth is the possible presence of Lavrentiev gap and related lack of regularity, non-density of smooth functions in the corresponding energy space. Finding assumptions for the presence of Lavrentiev phenomena is in particular important for regularity theory. We show that nonlocal and local-nonlocal models enjoy the presence of energy gap. We obtain the optimal conditions separating the regular case from the one with Lavrentiev gap for the different types of nonlocal and mixed local-nonlocal double phase models. The obtained conditions show the sharpness of resent regularity results for nonlocal double-phase problems.

Lavrentiev's phenomenon for double phase functionals

Michał Borowski
University of Warsaw, Poland

The talk shall introduce double phase functionals and spaces, and discuss their approximation properties in the context of the so-called Lavrentiev's phenomenon. The spaces of our interest are inhomogeneous Sobolev-type spaces lying between classical power-type Sobolev spaces. We shall see that regular functions are not always dense in such spaces, and discuss results on when we have the density. Then possible generalizations of those results to more general classes of problems shall be shown. This is joint work with Iwona Chlebicka, Filomena De Filippis, and Błażej Miasojedow.

Parabolic-like problems with $p(t, x)$ setting with discontinuity in time variable

Miroslav Bulíček
Charles University in Prague, Czech Republic

We consider a parabolic PDE with Dirichlet boundary condition and monotone operator A with non-standard growth controlled by an N-function depending on time and spatial variable. We do not assume continuity in time for the N-function. Using an additional regularization effect coming from the equation, we establish the existence of weak solutions and discuss also its uniqueness. We also extend the result to the generalized Stokes and Navier-Stokes problem with discontinuous in time rheology and also to the parabolic setting with only integrable data.

Sobolev embeddings in Musielak-Orlicz spaces

Andrea Cianchi
University of Florence, Italy

An embedding theorem for Sobolev spaces built upon general Musielak-Orlicz norms is offered. These norms are defined in terms of generalized Young functions which also depend on the x variable. Under minimal conditions on the latter dependence, a Sobolev conjugate is associated with any function of this type. Such a conjugate is sharp, in the sense that, for each fixed x, it agrees with the sharp Sobolev conjugate in classical Orlicz spaces. Both Sobolev inequalities in the whole $\mathbb{R}^n$ and mean-value Sobolev-Poincaré inequalities in bounded domains are established. In particular, optimal embeddings for standard Orlicz-Sobolev spaces, variable exponent Sobolev spaces, and double-phase Sobolev spaces are recovered as special instances. This is a joint work with Lars Diening.

Absence of Lavrentiev gap for functional with non standard growth

Filomena De Filippis
University of l'Aquila, Italy

We prove the absence of the Lavrentiev phenomenon for the following non-autonomous functional

$$\mathcal{F}(u) := \int_\Omega f(x, Du(x))dx,$$

where $f(x, z)$ satisfies a (p, q)-growth condition with respect to z and can be approximated by means of a suitable sequence of functions. We consider $B_R \Subset \Omega$ and the spaces

$$X = W^{1,p}(B_R, \mathbb{R}^N) \quad \text{and} \quad Y = W^{1,p}(B_R, \mathbb{R}^N) \cap W^{1,q}_{\text{loc}}(B_R, \mathbb{R}^N).$$

We also prove that the lower semicontinuous envelope $\bar{\mathcal{F}}_Y$ coincides with $\mathcal{F}$ or, in other words, that the Lavrentiev term is equal to zero for any admissible function $u \in W^{1,p}(B_R, \mathbb{R}^N)$.

Harmonic analysis in anisotropic generalized Orlicz spaces

Peter Hästö
University of Turku, Finland

Vector-valued generalized Orlicz spaces can be divided into anisotropic, quasi-isotropic and isotropic. In isotropic spaces, the Young function depends only on the length of the vector, i.e. $\Phi(v) = \phi(|v|)$. In the quasi-isotropic case $\Phi(v) \approx \phi(|v|)$ so the dependence is via the length of the vector up to a constant. In the anisotropic case, there is no such restriction, and the Young function depends directly on the vector. I will discuss recent advances on the boundedness of the maximal operator in the anisotropic case. I also summarize regularity results in the quasi-isotropic case, which include previous results as special cases.

Convexity, growth conditions and regularity

Lukas Koch
University of Max Planck, Germany

I will explain how growth conditions and convexity interact. This will lead to the introduction of a novel, but natural, type of growth condition called Fenchel (p, q)-growth or controlled duality (p, q)-growth. Using this growth condition, stronger regularity results than under controlled (p, q)-growth can be obtained. I will give examples of the statements that can be obtained. The talk is based on ongoing joint work with Cristiana de Filippis (Parma) and Jan Kristensen (Oxford).

Elliptic systems and double phase functionals

Francesco Leonetti
University of l'Aquila, Italy

It is well known that solutions to elliptic systems may be unbounded. Nevertheless, for some special classes of systems, it can be proved that solutions are bounded. We mention a recent result of this kind and we discuss some examples suggested by double phase functionals.

Interior regularity for $p,q-$PDEs with explicit $u-$dependence

Paolo Marcellini
University of Florence, Italy

We give some *existence* and *interior regularity results* for weak solutions of elliptic equations in divergence form of the type

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i} a^i \left(x, u\left(x\right), Du\left(x\right)\right) = b\left(x, u\left(x\right), Du\left(x\right)\right) \,,$$

in an open set $\Omega \subset \mathbb{R}^n$, $n \geq 2$. The vector field $\left(a^i\left(x,s,\xi\right)\right)_{i=1,2,\dots,n}$ satisfies some *general growth conditions* with respect to the gradient variable $\xi \in \mathbb{R}^n$, the so-called $p,q-$*growth conditions.* The novelties with respect to the mathematical literature on this topic are the general growth conditions and the explicit dependence of the differential equation on u, other than on its gradient Du and on the x variable.
Some details can be found in the recent articles:

P. Marcellini: Local Lipschitz continuity for $p,q-$PDEs with explicit $u-$dependence, Nonlinear Analysis, 226 (2023). https://doi.org/10.1016/j.na.2022.113066

G. Cupini, P. Marcellini, E. Mascolo: Local boundedness of weak solutions to elliptic equations with $p,q-$growth, Math. Eng., 5 (2023). https://doi.org/10.3934/mine.2023065

Lavrentiev's Phenomenon in Nonlinear Elasticity

Anastasia Molchanova
University of Vienna, Austria

In this talk, we present a new example of Lavrentiev's phenomenon in the context of nonlinear elasticity. This example is based on an interplay of the elastic energy's resistance to infinite compression and the Ciarlet–Necas condition, a constraint preventing global interpenetration of matter on sets of full measure.

The elliptic regularity problem beyond Uhlenbeck structure equations

Sunra Mosconi
University of Catania, Italy

I will describe some regularity results for minimisers of multiple integrals of the Calculus of Variation driven by integrands which are elliptic in a suitable sense introduced by Kovalev and Maldonado. This class of integrands turn out to be the closure with respect of a natural convergence of the classical strongly elliptic ones, resulting in a large class of integrands exhibiting general behaviour (including Uhlenbeck structure) and at the same time allowing degeneracy and/or singularity over possibly dense subsets of the gradient variable. On one hand this class of integrands provides a unified framework well beyond the classical degenerate/singular dichotomy; on the other, the irregular behaviour of their Hessian forces new techniques to tackle the regularity problem for the corresponding minimisers. The latter has been solved by Astala Iwaniec and Martin in the plane through complex variable methods but is completely open in higher dimensions. I'll discuss some partial results obtained in collaboration with U. Guarnotta (Un. Kore, Enna) and G. Marino (Augsburg Un.), highlighting if time permits some of the numerous open problems in this area.

Astala. K., Iwaniec, T. and Martin, G.J.: *Elliptic Partial Differential Equations and Quasiconformal Mappings in the Plane.* Princeton Mathematical Series **48**, Princeton University Press, 2009.

Guarnotta, U. and Mosconi, S.: A general notion of uniform ellipticity and the regularity of the stress field for elliptic equations in divergence form. *Anal. PDE* **16** (2023), 1955–1988.

Kovalev, L.V. and Maldonado, D.: Mappings with convex potentials and the quasiconformal Jacobian problem. *Illinois J. Math.* **49** (2005), 1039–1060.

Marino, M. and Mosconi, S.: Lipschitz regularity for solutions of a general class of elliptic equations. Preprint arXiv:2304.00657v2, (2023).

On the Lavrentiev phenomenon for vectorial, convex integral functionals

Matthias Ruf
Ecole polytechnique fédérale de Lausanne, Switzerland

The Lavrentiev phenomenon basically says that the infimum value of a functional may depend on the space of admissible functions (e.g., Sobolev, Lipschitz, smooth). In nonlinear elasticity the Lavrentiev phenomenon is sometimes considered as a possible onset of fracture or other effects such as cavitation. For convex autonomous integrands however, it is believed (and known in the scalar case) that no Lavrentiev phenomenon occurs at last for regular boundary conditions. In this talk we present some new results on the absence of the Lavrentiev phenomenon in the convex, possibly non-autonomous, setting. Joint work with Lukas Koch and Mathias Schäffner.

Non occurrence of the Lavrentiev gap for Dirichlet problems in quite general domains

Giulia Treu
University of Padova, Italy

We will present some recent results on the non occurrence of the Lavrentiev gap for multidimensional scalar functionals of the type

$$\int_{\Omega} f(x, u(x), \nabla u(x))\, dx \qquad u \in \varphi + W_0^{1,p}(\Omega)$$

where Ω is an open bounded domain in $\mathbb{R}^N$, $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ and φ is a Lipschitz function. In particular we will discuss conditions on the Lagrangian f and on the domain Ω that ensure that for every $u \in \varphi + W_0^{1,p}(\Omega)$ there exists a sequence $\{u_k\}_{k\in\mathbb{N}}, u_k \in \varphi + W_0^{1,\infty}(\Omega)$, such that u_k strongly converges in $W^{1,p}(\Omega)$ to u and

$$\lim_{k\to\infty} \int_{\Omega} f(x, u_k(x), \nabla u_k(x))\, dx = \int_{\Omega} f(x, u(x), \nabla u(x))\, dx$$

All the results are contained in joint papers with Pierre Bousquet (Université de Toulouse) and Carlo Mariconda (Università di Padova).
This is joint work with A. Cianchi, G. Ciraolo, A. Farina and V.G. Maz'ya.